I'll ignore normalization constants in the first bit.

Humbert: I can see that the operator acts much like the product rule for differentiation but I don't know why.

The reason is that there is subtlety here. There isn't one lowering operator, but rather two, one for each particle.

Lets call the angular momentum of the first particle J_1 and the angular momentum of the second particle J_2. The total angular momentum of the combine system is $J_{total} = J_1 + J_2$.

To lower the multiparticle state $\psi(\frac{3}{2}, \frac{3}{2})$ you need an operator which lowers the combined momentum J_{total}. As per usual I'll call the lowering operator J^{-}_{total}. Remembering that $J_{total} = J_1 + J_2$ then $J^{-}_{total} = J^{-}_1 + J^{-}_2$

Also $\psi(\frac{3}{2}, \frac{3}{2}) = \psi(1,1)\psi(\frac{1}{2}, \frac{1}{2})$

So we have:

$J^{-}_{total}\psi(\frac{3}{2}, \frac{3}{2}) = (J^{-}_1 + J^{-}_2)\psi(\frac{3}{2}, \frac{3}{2}) = (J^{-}_1 + J^{-}_2)\psi(1,1)\psi(\frac{1}{2}, \frac{1}{2}) = J^{-}_1\psi(1,1)\psi(\frac{1}{2}, \frac{1}{2}) + J^{-}_2\psi(1,1)\psi(\frac{1}{2}, \frac{1}{2})$

Now J_1 can only lower $\psi(1,1)$ because it only acts on the first particle and J_2 can only lower $\psi(\frac{1}{2}, \frac{1}{2})$ for the same reason.

Hence you have:
$J^{-}_1\psi(1,1)\psi(\frac{1}{2}, \frac{1}{2}) + J^{-}_2\psi(1,1)\psi(\frac{1}{2}, \frac{1}{2}) = \psi(1,0)\psi(\frac{1}{2}, \frac{1}{2}) + \psi(1,1)\psi(\frac{1}{2}, -\frac{1}{2})$

Humbert: giving a = sqrt(2/3) and b = -1/3 but I don't follow how.

Again for this question $J_{total} = J_1 + J_2$ is used. Now because $\psi(\frac{1}{2}, \frac{1}{2})$ is at the highest m value for its j value, J^{+}_{total} will annihilate it. Thus $J^{+}_{total}\psi(\frac{1}{2}, \frac{1}{2}) = 0$.

Again $J^{+}_{total} = J^{+}_1 + J^{+}_2$ and $\psi(\frac{1}{2}, \frac{1}{2}) = a(\psi(1,1)\psi(\frac{1}{2}, -\frac{1}{2})) + b(\psi(1,0)\psi(\frac{1}{2}, \frac{1}{2}))$

So $J^{+}_{total}\psi(\frac{1}{2}, \frac{1}{2}) = J^{+}_1 + J^{+}_2[a(\psi(1,1)\psi(\frac{1}{2}, -\frac{1}{2})) + b(\psi(1,0)\psi(\frac{1}{2}, \frac{1}{2}))] = 0$
This goes to:
$J^{+}_1[a(\psi(1,1)\psi(\frac{1}{2}, -\frac{1}{2})) + b(\psi(1,0)\psi(\frac{1}{2}, \frac{1}{2}))] + J^{+}_2[a(\psi(1,1)\psi(\frac{1}{2}, -\frac{1}{2})) + b(\psi(1,0)\psi(\frac{1}{2}, \frac{1}{2}))] = 0$

J^{+}_1 destroys $\psi(1,1)$ and raises $\psi(1,0)$ to $\psi(1,1)$ and J^{+}_2 destroys $\psi(\frac{1}{2}, \frac{1}{2})$ and raises $\psi(\frac{1}{2}, -\frac{1}{2})$ to $\psi(\frac{1}{2}, \frac{1}{2})$. So the equation above becomes:

$b(\psi(1,1)\psi(\frac{1}{2}, \frac{1}{2})) + a(\psi(1,1)\psi(\frac{1}{2}, \frac{1}{2})) = 0$
Hence for this to be true a = -b. Remember I haven't normalised things correctly when I used the raising operator. If you do that things should work out.